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EDGE EFFECT IN THE BENDING OF A THIN THREE-DIMENSIONAL PLATE*

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The boundary layer near the rigidly clamped edge of a thin three-dimensional plate subjected to bending loads is investigated. It is shown that taking account of the next term in the deflection asymptotic form results in the appearance of inhomogeneities in the boundary conditions on the plate edge. It is proved that far from the edge the difference in the solution of the problem in an invariant formulation and the three-dimensional solution is inversely proportional to the plate thickness (the error for the Kirchhoff solution is inversely proportional to the square of the thickness; near the edge the accuracies of both solutions is identical). A correction term is found in a representation of the eigenfrequencies of the bending vibrations and a comparison is made with the Reissner theory.

1. Formulation of the problem. Let Ω be a domain on the plane $\mathbf{R}^2$ bounded by a closed simple smooth (class C^∞) contour $\partial\Omega$, Q is a cylinder $\{x\colon y=(x_1,x_2)\in\Omega,\ |x_3|<{}^1/_2h\}$ of low altitude h with side surface S_h and bases $\Gamma_h{}^\pm$. We examine the three-dimensional problem of elasticity theory

$$\mu\Delta\mathbf{u}(h,\mathbf{x})+(\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u}(h,\mathbf{x})+h^{-1}f(y)\mathbf{e}^{(3)}=0,\quad \mathbf{x}\in Q_h \tag{1.1}$$

$$\boldsymbol{\sigma}^{(3)}(\mathbf{u};h,\mathbf{x})=p_\pm(\mathbf{y})\mathbf{e}^{(3)},\quad \mathbf{x}\in\Gamma_h{}^\pm \tag{1.2}$$

$$\mathbf{u}(h,\mathbf{x})=0,\quad \mathbf{x}\in S_h \tag{1.3}$$

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Here λ, μ are Lamé coefficients, $\mathbf{e}^{(j)}$ are directions in $\mathbf{R}^3$, $\mathbf{u} = (u_1, u_2, u_3)$ is the displacement vector, $p_\pm$ and $h^{-1}f$ are the transverse load and mass forces, $\sigma^{(3)} = (\sigma_{13}, \sigma_{23}, \sigma_{33})$, σ_{jk} are Cartesian components of the stress tensor $\sigma(\mathbf{u})$, $\delta_{j,k}$ is the Kronecker delta, and the subscript k after a comma denotes differentiation with respect to x_k. The characteristic dimension of the domain Ω is reduced to one by scaling. Then the small parameter h and the coordinates become dimensionless. We will find several of the first terms of the asymptotic form of the solution $\mathbf{u}$ of Problem (1.1)-(1.3) as $h \to 0$. We will use a modification of the well-known algorithm in /1-4/ to construct asymptotic expansions.

2. ***Internal solution.*** Far from S_h we will represent the solution in the form of the formal series

$$\mathbf{u}(h, \mathbf{x}) \sim \sum_{j=0}^{\infty} h^{j-3}(\mathbf{V}^j(\mathbf{y}) + h\mathbf{U}^j(\mathbf{y}, h^{-1}x_3)) \tag{2.1}$$
$$\mathbf{V}^q = \mathbf{e}^{(3)}w^q, \quad q = 0, 1; \quad \mathbf{V}^j = (\mathbf{v}^j, w^j), \quad \mathbf{v}^j = (v_1^j, v_2^j)$$

where the components $U_k^j(\mathbf{y}, \zeta)$ $(k = 1, 2, 3)$ have zero means $\langle U_k^j\rangle(\mathbf{y})$ in $\zeta \in (-1/2, 1/2)$; $\zeta = h^{-1}x_3$.

Let L and B be matrix differential operators from the left side of (1.1) and (1.2). It can be confirmed that they are represented as follows in the coordinates $\mathbf{y}, \zeta$

$$\begin{gathered} L(\partial/\partial\mathbf{x}) = h^{-2}L^\circ(\partial/\partial\zeta) - h^{-1}L^1(\partial/\partial\mathbf{y}, \partial/\partial\zeta) - L^2(\partial/\partial\mathbf{y}) \\ B(\partial/\partial\mathbf{x}) = h^{-1}B^\circ(\partial/\partial\zeta) - B^1(\partial/\partial\mathbf{y}) \\ L^\circ(\partial/\partial\zeta) = \mathrm{M}\partial^2/\partial\zeta^2, \quad B^\circ(\partial/\partial\zeta) = \mathrm{M}\partial/\partial\zeta, \quad \mathrm{M} = \operatorname{diag}(\mu, \mu, \lambda + 2\mu) \end{gathered} \tag{2.2}$$

We substitute (2.1) and (2.2) into (1.1) and (1.2) and collect coefficients of identical powers of h. We obtain recursion systems of equations with the parameter $\mathbf{y} \in \Omega$ to determine the vector functions $\mathbf{U}^j$

$$\begin{gathered} L^\circ\mathbf{U}^j = L^1\mathbf{U}^{j-1} + L^2(\mathbf{U}^{j-2} + \mathbf{V}^{j-1}) - \delta_{j,3}f\mathbf{e}^{(3)}, \quad \zeta \in (-1/2, 1/2) \\ B^\circ\mathbf{U}^j = B^1(\mathbf{U}^{j-1} + \mathbf{V}^j) + \delta_{j,3}p_\pm\mathbf{e}^{(3)}, \quad \zeta = \pm 1/2 \end{gathered} \tag{2.3}$$

Solving Problem (2.3), $j = 0, 1, 2,$ successively, we have

$$\begin{gathered} \mathbf{U}^\circ = -\zeta\boldsymbol{\nabla}w^\circ, \quad \mathbf{U}^1 = 1/2\,\nu(1-\nu)^{-1}(\zeta^2 - 1/12)\Delta_y w^\circ\mathbf{e}^{(3)} - \zeta\boldsymbol{\nabla}w^1 \\ \mathbf{U}^2 = 1/2(1-\nu)^{-1}\{1/3[\zeta^3(2-\nu) + 1/4(\nu-6)\zeta]\boldsymbol{\nabla}\Delta_y w^\circ + \\ \nu(\zeta^2 - 1/2)\Delta_y w^1\mathbf{e}^{(3)}\} - \zeta\boldsymbol{\nabla}w^2 \\ \boldsymbol{\nabla} = (\nabla, 0), \ \nabla = (\partial/\partial y_1, \partial/\partial y_2), \quad \Delta_x = \nabla\cdot\nabla \end{gathered} \tag{2.4}$$

where the dot denotes the scalar product. The condition for Problem (2.2) $(j = 3)$ to be solvable is the Sophie Germain equation

$$D\Delta_y^2 w^\circ(y) = p_+(y) - p_-(y) + f(y), \quad y \in \Omega; \quad D = E[12(1-\nu^2)]^{-1} \tag{2.5}$$

where D is the reduced $(h = 1)$ cylindrical stiffness of the plate, E is Young's modulus, and ν is Poisson's ratio. The solution $\mathbf{U}^3$ has the form

$$\begin{gathered} \mathbf{U}^3 = [4\mu(1-\nu)]^{-1}\{[1/2(3-\nu^2)(\zeta^2 - 1/12) - (1-\nu^2)(\zeta^4 - 1/80)(p_+ - \\ p_- + f) + (1-2\nu)[\zeta(p_+ + p_-) - (\zeta^2 - 1/12)f]\}\mathbf{e}^{(3)} + 1/6\zeta(1 - \\ \nu)^{-1}[\zeta^2(2-\nu) + 1/4(\nu-6)]\boldsymbol{\nabla}\Delta_y w^1 + \nu(1-\nu)^{-1}[1/2(\zeta^2 - \\ 1/12)\Delta_y w^2 - \zeta\nabla\cdot\mathbf{v}^3]\mathbf{e}^{(3)} \end{gathered} \tag{2.6}$$

In order for Problem (2.3) $(j = 4)$ to become solvable, three conditions must be satisfied. Firstly, there is the analogous equation to (2.5)

$$D\Delta_y^2 w^1(\mathbf{y}) = 0, \quad \mathbf{y} \in \Omega \tag{2.7}$$

and secondly, the system of two equations describing the generalized state of plane stress

$$\begin{gathered} \mu\Delta_y\mathbf{v}^3(\mathbf{y}) + \mu(1+\nu)(1-\nu)^{-1}\nabla\nabla\cdot\mathbf{v}^3(\mathbf{y}) = \\ 1/2\,\nu(1-\nu)^{-1}\nabla(p_+ + p_-)(\mathbf{y}), \quad \mathbf{y} \in \Omega \end{gathered} \tag{2.8}$$

3. ***The problem for a boundary layer.*** Series (2.1) does not satisfy Condition (1.3) on the cylinder side surface S_h. As we know, the residual that occurs is compensated by using a boundary layer. We introduce a set $\overline{S_h}$ of coordinates n, x_3, s in the neighbourhood of Υ where n and s are the internal normal and tangential components. We represent the solutions of boundary-layer type in the form $\mathbf{z} = (z_n, z_3, z_s)$. We set $\eta = (h^{-1}n, h^{-1}s)$. The operators L and B written in $n, x_3, s,$ coordinates can be split into the following formal series:

$$L\left(\frac{\partial}{\partial \mathbf{x}}\right) \sim \sum_{j=0}^{\infty} h^{j-2} P^j\left(\eta, s, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial s}\right), \tag{3.1}$$

$$B\left(\frac{\partial}{\partial \mathbf{x}}\right) \sim \sum_{j=0}^{\infty} h^{-1} Q^j\left(\eta, s, \frac{\partial}{\partial \eta}, \frac{\partial}{\partial s}\right)$$

It is clear that the operators P° and Q° do not contain differentiation with respect to s while their coefficients are constants. Moreover $P^\circ(\partial/\partial\eta) = L(\partial/\partial\eta, 0)$, $Q^\circ(\partial/\partial\eta) = B(\partial/\partial\eta, 0)$. Thus, a boundary value problem is obtained from (1.1)-(1.3) to determine the boundary layer

$$P^\circ(\partial/\partial\eta)\,\mathbf{z}(\eta) + \mathbf{H}(\eta) = 0, \quad \eta \in \Pi; \quad \mathbf{z}(0, \eta_2) = \mathbf{\Phi}(\eta_2), \quad |\eta_2| < {}^1/_2, \tag{3.2}$$
$$Q^\circ(\partial/\partial\eta)\,\mathbf{z}(\eta_1, \pm {}^1/_2) = \mathbf{G}^\pm(\eta_1), \; \eta_1 \in (0, \infty)$$

Problem (3.2) is considered in the half-strip $\Pi = \{\eta \in \mathbf{R}^2\colon \eta_1 > 0, \; |\eta_2| < {}^1/_2\}$ and is divided into two: the plane problem of elasticity theory (the first two lines) and the problem of antiplane shear (the third line). Moreover, its data depend parametrically on the variable s; the symbol s is omitted to shorten the notation.

The following assertions are essentially known and are obtained by making the general results specific /5, 6/. (The calculations needed for this can be found in /4/, say; see /7, 8/ also).

Proposition 1. 1°. Let the right-hand sides of (3.2) decrease exponentially as $\eta_1 \to +\infty$. Then a unique solution $\mathbf{z}$ exists for Problem (3.2) with bounded components z_n and z_s. The asymptotic expansion

$$\mathbf{z}(\eta) = \mathbf{C} + (-\eta_2 C_4, \eta_1 C_4, 0) + o(\exp(-\delta\eta_1)), \quad \eta_1 \to +\infty \tag{3.3}$$

holds for it for a certain $\delta > 0$.

2°. Exactly four linearly independent solutions exist

$$\mathbf{Y}^{(j)}(\eta) = \mathbf{\Psi}^{(j)}(\eta) + \mathbf{X}^{(j)}(\eta), \quad j = 1, 2, 3, 4 \tag{3.4}$$

for the homogeneous Problem (3.2) that grow at infinity not more rapidly than a power of η_1. The $\mathbf{X}^{(j)}$ in (3.4) are subject to Conditons 1° of the solution of Problem (3.2) with right-hand sides $\mathbf{H} = 0$, $\mathbf{G}^\pm = 0$, $\mathbf{\Phi}(\eta_2) = -\mathbf{\Psi}^{(j)}(0, \eta_2)$ and the vectors $\mathbf{\Psi}^{(j)}$ are given by the inequalities

$$\mathbf{\Psi}^{(1)}(\eta) = \left(\eta_1, -\frac{\nu\eta_2}{1-\nu}, 0\right), \tag{3.5}$$

$$\mathbf{\Psi}^{(2)}(\eta) = \left(\frac{1}{2}\frac{\eta_1^2}{2} + \frac{2-\nu}{1-\nu}\frac{\eta_2^3}{6} + \frac{\eta_2(\nu-6)}{24(1-\nu)}, \frac{\eta_1^3}{6} + \frac{\nu\eta_1}{2(1-\nu)}\left(\eta_2^2 - \frac{1}{12}\right), 0\right),$$

$$\mathbf{\Psi}^{(3)}(\eta) = (0, 0, \eta_1), \quad \mathbf{\Psi}^{(4)}(\eta) = \left(-\eta_1\eta_2, \frac{1}{2}\left(\eta_1^2 + \frac{\nu}{1-\nu}\left(\eta_2^2 - \frac{1}{12}\right)\right), 0\right)$$

3°. The constants $\mathbf{C} = (C_1, C_2, C_3)$ and C_4 from (3.3) are evaluated from the formulas

$$C_j = \alpha_j \Lambda(\mathbf{G}^\pm, \mathbf{H}, \mathbf{\Phi}; \mathbf{Y}^{(j)}), \quad j = 1, 2, 4 \tag{3.6}$$

$$C_3 = \mu^{-1}\int_\Pi \eta_1 H_3(\eta)\,d\eta + \int_{-1/2}^{1/2} \Phi_3(\eta_2)\,d\eta_2 + \mu^{-1}\sum_\pm \pm\int_0^\infty \eta_1 G_3^\pm(\eta_1)\,d\eta_1$$

$$\alpha_1 = (12D)^{-1}, \quad \alpha_4 = -\alpha_2 = D^{-1},$$

$$\Lambda(\mathbf{G}^\pm, \mathbf{H}, \mathbf{\Phi}; \mathbf{Y}) = \int_\Pi \mathbf{Y}'(\eta)\cdot\mathbf{H}(\eta)\,d\eta + \int_{-1/2}^{1/2} \mathbf{T}^{(1)}(\mathbf{Y}; 0, \eta_2)\cdot\mathbf{\Phi}(\eta_2)\,d\eta_2 +$$

$$\sum_\pm \pm\int_0^\infty \mathbf{Y}'(\eta_1, \pm {}^1/_2)\cdot\mathbf{G}^\pm(\eta_1)\,d\eta_1, \quad \mathbf{Y}' = (Y_1, Y_2, 0)$$

Applying (3.3) and (3.6) to the solution (3.4), $j = 4$ we arrive at the following assertion.

Proposition 2. The following relationships hold

$$\mathbf{X}^{(4)}(\eta) = (c(\nu)\eta_2,\ b(\nu) - c(\nu)\eta_1,\ 0) + o(\exp(-\delta\eta_1)),\quad \eta_1 \to +\infty \tag{3.7}$$
$$c(\nu) = D^{-1}\mathrm{Э}(\mathbf{X}^{(4)},\ \mathbf{X}^{(4)};\ \Pi)$$
$$\mathrm{Э}(\mathbf{X},\mathbf{Y};\Pi) = E^{-1}(1+\nu)\int_{\Pi}\sum_{j,k=1}^{2}[T_{jk}(\mathbf{X})T_{jk}(\mathbf{Y}) - \nu T_{jj}(\mathbf{X})T_{kk}(\mathbf{Y})]\,d\eta$$
$$\mathbf{T}^{(1)} = (T_{11}, T_{12}, 0);\quad T_{jk}(\mathbf{z}) = \mu(\partial z_k/\partial\eta_j + \partial z_j/\partial\eta_k) + \delta_{j,k}\lambda\nabla_\eta\cdot\mathbf{z}$$
$$T_{j3}(\mathbf{z}) = T_{3j}(\mathbf{z}) = \mu\partial z_3/\partial\eta_j,\quad T_{33}(\mathbf{z}) = \lambda\nabla_\eta\cdot\mathbf{z},\quad j,k = 1,2$$

The dependence of the quantity $c(\nu)$ on Poisson's ratio determined by using computer computations is represented in Fig.1.

4. The boundary conditions on $\partial\Omega$. In order to compensate the residual of series (2.1) in the boundary Condition (1.3), we shall seek the boundary layer in the form

$$\mathbf{z}(h, x) \sim \sum_{j=0}^{\infty} h^{j-3}\mathbf{z}^j(h^{-1}n, h^{-1}x_3, s) \tag{4.1}$$

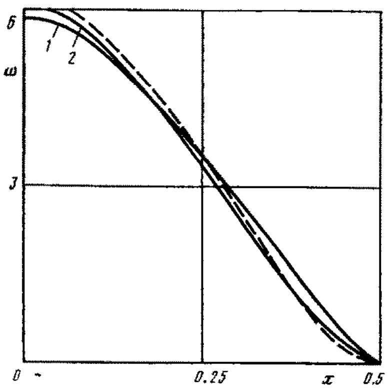

Fig.1

where $\eta \to \mathbf{z}^j(\eta, s)$ is a vector function decreasing exponentially as $\eta_1 \to +\infty$. To the accuracy $O(h^{-1})$ the residual mentioned equals $h^{-3}(0, w^\circ(s,0), 0) + h^{-2}(\eta_2 w^\circ_{,n}(s, 0), -w^1(s, 0), \eta_2 w^\circ_{,s}(s, 0))$ on S_h in the coordinates η, s. According to Proposition 1, 3° Problem (3.2) with appropriate right-hand sides has solutions $\mathbf{z}^\circ$ and $\mathbf{z}^1$ that vanish at infinity only if the following equalities are satisfied:

$$w^\circ(\mathbf{y}) = 0,\quad (\partial w^\circ/\partial n)(\mathbf{y}) = 0,\quad \mathbf{y} \in \partial\Omega \tag{4.2}$$
$$w^1(\mathbf{y}) = 0,\quad \mathbf{y} \in \partial\Omega \tag{4.3}$$

Here $\mathbf{z}^\circ = \mathbf{z}^1 = 0$. Furthermore, the vector $\mathbf{z}^2$ is a solution of Problem (3.2) in which $\mathbf{H} = 0$, $\mathbf{G}^\pm = 0$ and

$$\Psi(\eta_2) = (\eta_2 w^1_{,n}(s,0),\ -{}^1/_2\nu(1-\nu)^{-1}(\eta_2{}^2 - {}^1/_{12})\Delta_y w^\circ(s,0) - w^2(s,0),\ 0) \tag{4.4}$$

Comparing (4.4) with the boundary Conditions (3.5) mentioned in Proposition 1 for the solution $\mathbf{X}^{(4)}$, we find that

$$\mathbf{z}^2(\eta, s) = w^1_{,n}(s,0)(\eta_2\mathbf{e}^{(n)} - \eta_1\mathbf{e}^{(3)}) - \mathbf{e}^{(3)}w^2(s,0) + \mathbf{X}^{(4)}(\eta)\Delta_y w^\circ(s,0) \tag{4.5}$$

Applying Proposition 2, we obtain that the requirement of an exponential decrease in the vector (4.5) yields the equality

$$w^1_{,n}(\mathbf{y}) + c(\nu)\Delta_y w^\circ(\mathbf{y}) = 0,\quad \mathbf{y} \in \partial\Omega \tag{4.6}$$
$$w^2(\mathbf{y}) = b(\nu)\Delta_y w^\circ(\mathbf{y}),\quad \mathbf{y} \in \partial\Omega \tag{4.7}$$

The relationships (4.2), (4.3) and (4.6) are the necessary boundary conditions for Eqs. (2.5) and (2.7).

5. Justification of the asymptotic expansion. We will first find the first terms of the asymptotic form of the stress tensor components that are generated by the sum of the series (2.1) and (4.1). According to (2.4) and (2.6), far from the side surface S_h we set

$$\sigma^*_{jk} = -12D\zeta h^{-2}[\nu\delta_{j,k}\Delta_y + (1-\nu)\partial^2/\partial y_j\partial y_k](w^\circ + hw^1) \tag{5.1}$$
$$\sigma^*_{j3} = 6Dh^{-1}(\zeta^2 - {}^1/_4)(\partial/\partial y_j)\Delta_y(w^\circ + hw^1),\quad j,k = 1,2$$
$$\sigma^*_{33} = {}^1/_2(p_+ + p_- - 2\zeta f + \zeta(3 - 4\zeta^2)(p_+ - p_- + f))$$

The boundary layer yields an additional stress field near the plate edge. The appropriate tensor σ^z written in n, x_3, s, coordinates is calculated in the displacements $h^{-1}\chi(\mathbf{z}^2 + b(\nu)\Delta_y w^\circ\mathbf{e}^{(3)})$ (the reasons for introducing the component $b(\nu)\Delta_y w^\circ\mathbf{e}^{(3)}$ are clarified before Proposition 4) by using the usual formulas for the state of plane strain and equals

$$\| T_{p,q}(h^{-1}\chi[\mathbf{z}^2 + \mathbf{e}^{(3)} b(\nu) \Delta_y w^\circ]) \|_{p,q=1}^3$$

where χ is a smooth shearing function, and the quantities T_{jk} are mentioned in Proposition 1, 3°. We emphasize that by virtue of Proposition 2 the residuals occurring because of multiplication by the cutoff $\chi(n)$ are exponentially small.

The displacement vector $\mathbf{u}^\circ = h^{-3}\mathbf{e}^{(3)} w^\circ + h^{-2}\mathbf{U}^\circ$ (see (2.1) and (2.4)) and the stress vector σ° (terms of the order of $O(h^{-1})$ and $O(1)$, respectively, are discarded in the first two formulas in (5.1)), are called the fundamental approximation to the solution of Problem (1.1)-(1.3). By analogy the displacement $\mathbf{u}^1 = h^{-3}\mathbf{e}^{(3)}(w^\circ + hw^1) + h^{-2}(\mathbf{U}^\circ + h\mathbf{U}^1) + \chi h^{-1}(\mathbf{z}^2 + b(\nu)\Delta_y w^\circ \mathbf{e}^{(3)})$ and the stress $\sigma^1 = \sigma^* + \sigma^z$ should be considered to be second-order approximations. The latter is needed in the foundation. An assertion concerning the fundamental term of the asymptotic form is known /9, 10/.

Proposition 3. Let $f, p_\pm \in L_2(\Omega)$. Then the inequality

$$\|(d+h)^{-2}(u_3 - u_3^\circ)\| + h^{-1}\|(d+h)^{-1}(u_j - u_j^\circ)\| + \|(d+h)^{-1}\nabla_y(u_3 - u_3^\circ)\| + \tag{5.2}$$
$$h^{-1}\|\nabla(u_j - u_j^\circ)\| + h\|(d+h)^{-2}(\partial/\partial x_3)(u_3 - u_3^\circ)\| +$$
$$\|(d+h)^{-1}(\partial/\partial x_3)(u_j - u_j^\circ)\| + h^{-1}\|\sigma_{jk}(\mathbf{u}) - \overset{\circ}{\sigma}_{jk}\| +$$
$$\|(d+h)^{-1}(\sigma_{j3}(\mathbf{u}) - \sigma_{j3}^\circ)\| + h^{-1}\|\sigma_{33}(\mathbf{u}) - \sigma_{33}^\circ\| \leqslant ch^{-2}\Xi_0$$

holds for the solution of Problem (1.1)-(1.3).

The norm in $L_2(Q_h)$ is here calculated throughout; the subscripts j, k take the values 1, 2; $d(\mathbf{y})$ is the distance between the point $\mathbf{y}$ and $\partial\Omega$ the constant c is independent of h; Ξ_α is the sum of the norms of the quantities $f, p_\pm$ in $W_2^\alpha(\Omega)$, $\alpha = 0,1$.

If $f, p_\pm \in W_2^1(\Omega)$ we obtain from (2.5), (2.2) and (2.7), (4.3) and (4.6) that $w^\circ \in W_2^5(\Omega)$, $w^1 \in W_2^4(\Omega)$, $\Delta w^\circ \in W_2^3(\Omega)$. Consequently, the second approximation possesses the requisite smoothness. Moreover, the errors left in (1.1)-(1.3) have the structure of the errors of the first approximation, and their order is one higher as $h \to 0$. We especially emphasize that the boundary Condition (1.3) is satisfied because of the presence of the component $\chi h^{-1} b(\nu)\Delta_y w^\circ$ in u_3^1. This expression depends smoothly on the variable $y \in \bar{\Omega}$, its residual in (1.1) and (1.2) is small, elimination of such a non-decreasing term of the boundary layer occurs at the next step of the algorithm (see (4.7)). The following assertion results from the above.

Proposition 4. Let $f, p_\pm \in W_2^1(\Omega)$. If the subtrahend on the left in (5.2) is replaced by the second approximation to the solution of Problem (1.1)-(1.3), then the majorant takes the form $c_1 h^{-1}\Xi_1$.

In particular, Proposition 4 shows that the estimate (5.2) is not improvable and asymptotically exact since the norms on the left in (5.2) calculated from the boundary layer are of order h^{-2}.

We will now turn our attention to the following important fact. Propositions 5 and 4 yield an estimate of the asymptotic approximations only in the mean (in an energetic metric). Meanwhile by using local estimates /11/ of the solutions of general elliptic boundary value problems estimates can be deduced /12/ of solutions of problems (1.1)-(1.3) in the Hölder spaces $C^{l,\alpha}(Q_h \setminus \Upsilon_h)$, Υ_h are small neighbourhoods of singular lines-edges $\partial\Omega \times \{\pm^1/_2 h\}$. Unfortunately, the inequalities obtained in this manner are unsatisfactory: because of the smallness of the "thickness" of the domain Q_h the estimates mentioned contain the parameter h and the constant $c(h)$ in the final inequality has a power growth as $h \to 0$. The possibility of constructing smaller terms of the asymptotic form eliminates this disadvantage. It is sufficient to take a certain number of "superfluous" terms of the series, to use a rough estimate in the Hölder spaces, and then to refer them to the remainder by evaluating the norms of these terms directly. We emphasize that the results /13/ on elliptical boundary value problems with ribs on the boundary enable the estimate to be extended to the whole cylinder Q_h.

We present an inequality for the second-order approximation under the assumption that the right-hand sides in (1.1) and (1.2) possess sufficient smoothness

$$|u_3 - u_3^1| + h^{-1}\{|u_j - u_j^1| + d_*|\sigma(\mathbf{u}) - \sigma^1|\} \leqslant \mathrm{c}h^{-1} \tag{5.3}$$

Here the displacements and stresses are calculated at any point $x \in \bar{Q}_h$; $d_*(x)$ is the distance between x and the rib $\partial\Omega \times \{\pm^1/_2 h\}$. We note that the estimate (5.3) remains valid if the displacement u_3^1 is determined without taking the boundary layer into account.

An estimate of the closeness of the vector $\mathbf{u}^* = h^{-3}\mathbf{e}^{(3)}(w^\circ + hw^1) + h^2(\mathbf{U}^\circ + h\mathbf{U}^1)$ and the tensor σ^* to the solution of Problem (1.1)-(1.3) is needed later. Since the boundary layer is concentrated in a small neighbourhood of the side surface S_h, its influence can be eliminated because of the introduction of additional weighting factors $d+h$ into the norm from the left side of (5.2). Therefore, by using the estimate the following assertion is obtained from Proposition 4.

Proposition 5. Let $f, p_\pm \in W_2^1(\Omega)$. The following inequality holds:

$$\begin{gathered}\|(d+h)^{-1}(u_3-u_3^*)\| + h^{-1}\|u_j-u_j^*\| + \|\nabla_y(u_3-u_3^*)\| + \\ h^{-1}\|(d+h)\nabla(u_j-u_j^*)\| + h\|(d+h)^{-1}(\partial/\partial x_3)(u_3-u_3^*)\| + \\ \|(\partial/\partial x_3)(u_j-u_j^*)\| + h^{-1}\|(d+h)(\sigma_{jk}(\mathbf{u})-\sigma_{jk}^*)\| + \|\sigma_{j3}(\mathbf{u})-\sigma_{j3}^*\| + \\ h^{-1}\|(d+h)(\sigma_{33}(\mathbf{u})-\sigma_{33}^*)\| \leqslant ch^{-1}\Xi_1\end{gathered} \tag{5.4}$$

We note that the estimates (5.2) and (5.4) are identically accurate in a *Ch* - neighbourhood of S_h, but the estimate (5.4) is more exact by an order far from the side surface of the cylinder Q_h (in the zone $d+h \sim 1$).

*6. **The combined problem.*** The scalars w° and w^1 are solutions of the same kinds of Problems (2.5), (4.2) and (2.7), (4.3), (4.6). Consequently, it is natural to combine their boundary conditions by formulating a new boundary value problem whose solution will coincide with the sum $w^\circ + hw^1$ to $O(h^2)$ accuracy. The simplest such problem is

$$\begin{gathered}D\Delta_y^2 w^*(h,\mathbf{y}) = p_+(\mathbf{y}) - p_-(\mathbf{y}) + f(\mathbf{y}), \quad \mathbf{y}\in\Omega \\ w^*(h,\mathbf{y}) = 0, \quad (\partial w^*/\partial n)(h,\mathbf{y}) + c(\nu)h\Delta_y w^*(h,\mathbf{y}) = 0, \quad \mathbf{y}\in\partial\Omega\end{gathered} \tag{6.1}$$

However, the energy functional corresponding to (6.1) is positive-definite just for $\nu = 0$ (for $\nu > 0$ the constant $c(\nu)$ is positive because of (3.7)). To construct the combined boundary value problem with a positive energy functional, we introduce a regularly perturbed domain $\Omega_h = \{\mathbf{y}\in\Omega\colon n > c(\nu)h\} \subset \Omega$. In other words, we remove a boundary strip of small width from the domain Ω in which the influence of plate edge clamping is essential. We consider the problem

$$D\Delta_y^2 w^*(h,\mathbf{y}) = p_+(\mathbf{y}) - p_-(\mathbf{y}) + f(\mathbf{y}), \quad \mathbf{y}\in\Omega_h \tag{6.2}$$

$$w^*(h,\mathbf{y}) = (\partial w^*/\partial n)(h,\mathbf{y}) = 0, \quad \mathbf{y}\in\partial\Omega_h \tag{6.3}$$

in Ω_h

Proposition 6. If $f, p_\pm \in W_2^1(\Omega)$, then the estimate

$$\| w^* - w^\circ - hw^1;\ W_2^3(\Omega_n)\| \leqslant ch^2$$

is true for the solution of Problem (6.2) and (6.3) where w° and w^1 satisfy Conditions (2.5), (4.2) and (2.7), (4.3), and (4.6).

Proof. We expand w° and w^1 in Taylor series in the variable n and substitute the sum $w^\circ + hw^1$ into the boundary Condition (6.3). For $n = c(\nu)h$ we have

$$\begin{gathered}w^\circ(\mathbf{y}) + hw^1(\mathbf{y}) = hw^1(0,s) + O(h^2) \\ \frac{\partial w^\circ}{\partial n}(\mathbf{y}) + h\frac{\partial w^1}{\partial n}(\mathbf{y}) = h\left(\frac{\partial w^1}{\partial n}(0,s) + c(\nu)\frac{\partial^2 w^\circ}{\partial n^2}(0,s)\right) + O(h^2)\end{gathered} \tag{6.4}$$

Since $\partial^2 w^\circ/\partial n^2 = \Delta w^\circ$ by virtue of (4.2) on $\partial\Omega$, it follows from (4.3) and (4.6) that the right-hand sides are of the order of h^2. It remains to use the estimates of the solutions of the elliptical problems (since the perturbation of the domain is regular then, according to /14, 15/ the constant in the appropriate inequality is independent of h).

As in the case of (2.4) we set

$$\mathbf{U}^* = h^{-3}\mathbf{e}^{(3)}[w^* + {}^1/_{12}h^2\nu(1-\nu)^{-1}(\zeta^2 - {}^1/_{12})\Delta_y w^*] - h^{-2}\xi\nabla w^* \tag{6.5}$$

and use the notation $\Sigma_{jk}^*, \Sigma_{j3}^*, \Sigma_{33}^*$ for the stresses calculated by means of (5.1) in which the sum $w^\circ + hw^1$ is replaced by w^*. There results from Propositions 5 and 6

Proposition 7. If $f, p_\pm \in W_2^1(\Omega)$, then the inequality (5.4) is true in which the quantities $\mathbf{U}^*, \mathbf{\Sigma}^*$ take part in place of $\mathbf{u}^*, \sigma^*$.

*7. **Example.*** To estimate the accuracy of the approximate formulas we computed the problem of the bending of a beam clamped at the endfaces by a constant transverse load on a computer by using two-dimensional finite elements, for $\nu = 0.3$, $h = 0.2$, $\Omega = (-{}^1/_2, {}^1/_2)$. The binomial asymptotic expansions of the beam deflection determined according to Propositions 6 and 4 are identical to $O(h^{-1})$ accuracy and are given by the equalities

$$w^*(x, h) = 2pD^{-1}h^{-3}(x^2 - (c(\nu)h - {}^1/_2)^2)^2 \tag{7.1}$$
$$h^{-3}w_0(x) + h^{-2}w_1(x) = 2pD^{-1}h^{-3}(x^2 - {}^1/_4)(x^2 + 2c(\nu)h - {}^1/_4)$$

(dimensionless coordinates). Graphs of the computer computations (curve *1*), the function (7.1) (curve *2*), and the Kirchhoff solution (dashes) are represented in Fig.2 $\omega = (2p)^{-1}wD$. We note that the relative error in the calculations of the normal displacements at the middle part of the beam is 5-6% by the Kirchhoff theory (as compared with the numerical solution) and does not exceed 1-2% when using (7.1).

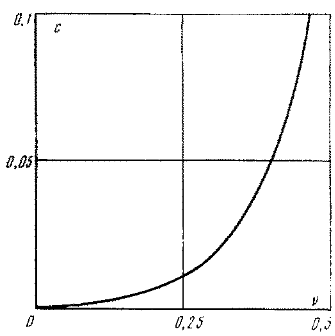

Fig.2

8. Plate bending vibrations. According to Proposition 7, the solution of Problem (6.2) and (6.3) reaches an asymptotic approximation of increased accuracy far from the edge S_h of the plate Q_h. It is natural to use this circumstance to find the refined eigenfrequency asymptotic form. The system of equations

$$\mu\Delta\mathbf{u}(h, \mathbf{x}) + (\lambda + \mu)\,\mathrm{grad\,div}\,\mathbf{u}(h, \mathbf{x}) + \rho k(h)^2\mathbf{u}(h, \mathbf{x}) = 0, \quad \mathbf{x} \in Q_h \tag{8.1}$$

with homogeneous boundary Conditions (1.2) and (1.3) (under the assumption that u_3 is even and u_1, u_2 are odd functions of the variable x_3) describes plate bending vibrations. It is known that the solutions of the spectral equation

$$D\Delta_y^2 w^\circ(\mathbf{y}) = \rho k_0^2 w^\circ(\mathbf{y}), \mathbf{y} \in \Omega \tag{8.2}$$

with boundary Conditions (4.2) yield the asymptotic form $k^{(n)}(h) = hk_0^{(n)} + O_n(h^2)$ of the eigenfrequencies of such vibrations (the estimate of the remainder is worsened as the number $n = 1, 2, \ldots$ increases). Combining the results obtained in the previous sections with the known procedure for constructing the asymptotic form of the eigenvalues of singularly perturbed elliptical boundary value problems (see /6/, Ch.2, say), we obtain that the solution of the spectral equation

$$D\Delta_y^2 w^*(\mathbf{y}, h) = \rho k_*(h)^2 w^*(\mathbf{y}, h), \quad \mathbf{y} \in \Omega_h \tag{8.3}$$

with Conditions (6.3) on $\partial\Omega_h$ yields an approximation to $k(h)$ with increased accuracy $O_n(h^3)$. We will formulate the appropriate assertion.

Proposition 8. The reltionships $k^{(n)}(h) = hk_*^{(n)}(h) + O(h^3)$ hold, where $n = 1, 2, \ldots$, $k^{(n)}(h)$ and $k_*^{(n)}(h)$ are eigenfrequencies of Problems (8.1), (1.2), (1.3) and (8.3), (4.2) arranged in non-decreasing order (taking multiplicity into account).

We mention still another asymptotic representation for $k(h)$. We assume that $\Lambda_0 = D^{-1}\rho k_0^3$ is a simple eigenvalue of the Dirichlet problem for the biharmonic operator and we normalize its corresponding eigenfunction in $L_2(\Omega)$. Repeating the computations from Sects.2 and 4, we obtain that the quantity Λ_1 in the representation

$$\Lambda(h) = h^2(\Lambda_0 + h\Lambda_1 + O(h^2)) \tag{8.4}$$

for the eigenvalue of the operator of problem (1.1)-(1.3) is found when solving the equation $\Delta_y^2 w^1 = \Lambda_0 w^1 + \Lambda_1 w^\circ$ with the boundary Conditions (4.3) and (4.6). By using Green's formula the conditions for such a problem to be solvable are reduced to the form

$$\Lambda_1 = c(\nu)\int_{\partial\Omega} |\Delta_y w^\circ(\mathbf{y})|^2\, ds \tag{8.5}$$

Therefore, the relationship $k(h) = hk_0(1 + {}^1/_2\rho^{-1}Dh\Lambda_1 k_0^{-2} + O(h^2))$ holds. See /16/ for its proof. We recall that D is the reduced cylindrical stiffness of the plate $(h = 1)$. By virtue of (3.7) and (8.4) Λ_1 is a non-negative quantity. Therefore, the eigenfrequencies in Problem (8.2), (4.2) yield an approximation to the eigenfrequencies of the bending vibrations of the plate Q_h with a disadvantage.

We present the expansions of the eigenfrequency $k_R(h) = [D\rho^{-1}\Lambda_R(h)]^{1/2}$ of the Reissner plate bending vibrations

$$\Lambda_R(h) = h^2\left\{\Lambda_0 + h^2\left({}^1/_{12}\Lambda_0\int_\Omega |\nabla w^\circ|^2\, d\mathbf{y} - [5(1-\nu)]^{-1}\int_\Omega |\nabla\Delta w^\circ|^2\, d\mathbf{y}\right) + O(h^3)\right\} \tag{8.6}$$

where Λ_0 and w° are the same as in (8.4) and (8.5). (Formula (8.6) can be obtained by using the results in /17/ when considering the Reissner plate theory problem as a singular perturbation of the Kirchhoff theory problem). Comparing (8.4), (8.5) and (8.6) we see that the correction h^3k_{1R} of the eigenfrequency by Reissner theory differs in order and is not sign-definite.

Formula (8.4) contradicts the inequality $k(h) \leqslant k_0(h)$ that is evident at first glance. The appropriate "proof" for the first frequencies is naturally carried out by using the Rayleigh principle

$$\rho k(h)^2 = \inf \{2Э(\mathbf{u}) \| \mathbf{u}; L_2(Q_h)\|^{-2} \mid \mathbf{u} \in W_2^1(Q_h), \mathbf{u} = 0 \quad \text{on} \quad S_h\} \tag{8.7}$$

$$\rho k_0^2 h^2 = \inf \{2h^2 Э_K(w) \| w; L_2(\Omega)\|^{-2} \mid w \in W_2^2(\Omega), w = \partial w/\partial n \quad \text{on} \quad \partial\Omega\} \tag{8.8}$$

$$Э_K(w) = {}^1/_2 D \, \| \Delta w; \; L_2(\Omega) \|^2$$

($Э(\mathbf{u})$ is the three-dimensional plate strain energy). The vector field of the displacement $\mathbf{u}^w = (-x_3\nabla_y w, w) \in W_2^1(Q_h)$, is constructed by means of the function w, and satisfies the boundary conditions on the plate side surface and seemingly $\inf \{2Э(\mathbf{u}) \| \mathbf{u} \|^{-2}\}$ in every case is not less than the right-hand side of (8.7) in a narrower set $\{\mathbf{u}^w\}$. The error is that $Э(\mathbf{u}^w) = (1 + \delta) h^3 Э_K(w)$, $\delta = \nu^2(1 - 2\nu)^{-1}$, and the infimum of the quantity $2h^2(1 + \delta) Э_K(w) (\| w \|^2 + {}^1/_{12} h^2 \| \nabla w \|^2)^{-1}$ must be sought in the set $\{\mathbf{u}^w\}$. Therefore, application of the Rayleigh principle does not yield on a priori estimate for the eigenfrequency $k(h)$.

We note that because of the smallness of the quantity $c(\nu)$ the correction term $O(c(\nu) h)$ can be less than the next term of the asymptotic form $O(h^2)$. Consequently, the correction found is decisive in the asymptotic form only for thin plates (according to computer computations for $h \leqslant 1/20$).

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